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On the Two-Dimensional Coulomb Gas

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This is a sequel to a recent work of Gaudin, who studied the classical equilibrium statistical mechanics of the two-dimensional Coulomb gas on a lattice at a special value of the coupling constant Γ such that the model is exactly solvable. This model is briefly reviewed, and it is shown that the correlation functions obey the sum rules that characterize a conductive phase. A related model in which the particles are constrained to move on an array of equidistant parallel lines has simpler mathematics, and the asymptotic behavior of its correlation functions is studied in some detail. In the low-density limit, the lattice model is expected to have the same properties as a system of charged, hard disks; the correlation functions, internal energy, and specific heat of the latter are discussed.

KEY WORDS: Coulomb systems; solvable models; correlations; sum rules; charged, hard disks.

1. INTRODUCTION

A considerable amount of work has already been done on the equilibrium statistical mechanics of the two-dimensional Coulomb gas with logarithmic interactions: the interaction potential between two particles as a function of their distance r is $\pm e^2 \ln(r/a)$, where a is some arbitrary length scale, which fixes the zero of energy. At the inverse temperature β the dimensionless coupling constant is $\Gamma = \beta e^2$. In this paper we consider a two-component plasma; then the attraction between oppositely charged particles competes with the thermal motion and gives rise to two kinds of phenomena that do not occur in a one-component plasma. In the first place, at short distances this attraction makes the partition function diverge when $\Gamma > 2$: the system

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becomes unstable against the collapse of pairs of oppositely charged particles, so that the point-particle model[1] is well-behaved only for $\Gamma < 2$. However, if the collapse is avoided by some short-range repulsion (hard cores, for instance), the model remains defined for lower temperatures. Then, for $\Gamma > 4$ the long range of the Coulomb attraction binds positive and negative particles in pairs of finite polarizability. Thus, at some critical value $\Gamma_c \sim 4$ the system undergoes the Kosterlitz–Thouless[2] transition between a high-temperature ($\Gamma < 4$) conductive phase and a low-temperature ($\Gamma > 4$) dielectric phase. In the present paper, which is about the Coulomb gas at $\Gamma = 2$ (a value for which the system is believed to be in a conductive phase), the Kosterlitz–Thouless transition will not be discussed further.

It has been known for some time that the two-dimensional Coulomb gas is equivalent to a Fermi field model,[3–5] which turns out to be a free one when $\Gamma = 2$. However, the necessity of introducing a short-range repulsion causes difficulties. In a recent paper, Gaudin[6] was able to circumvent these difficulties by studying a lattice version of the Coulomb gas, which he devised to be exactly solvable at $\Gamma = 2$.

Since at $\Gamma = 2$ we have at hand a solvable model, it is worth to study it further. The present paper is a sequel to Gaudin's work. In Section 2, we review his lattice model and supplement his results on several points. In Section 3, we consider a related model of charged point particles moving along an array of equidistant parallel lines. In Section 4, we study the continuum limit (small lattice spacing) and apply the results to a system of charged, hard disks at low density.

In Appendix A, we describe a modified Debye–Hückel theory, which is valid for the lattice model in the weak coupling limit $\Gamma \to 0$. In Appendix B, we revisit the equivalence of the Coulomb gas and free Fermi field models, at $\Gamma = 2$.

2. THE GAUDIN MODEL: REVIEW AND COMPLEMENTS

2.1. Definition of the Model

In a lattice model no collapse can occur. In order to handle the charges according to their positions in an easily tractable way, Gaudin[6] introduced two interwoven sublattices X and Y, arranged as shown in Fig. 1. Positive (negative) particles of charge e $(-e)$ sit on the sublattice X (Y); each site is occupied by no or one particle. The position of the ith site is defined by the complex number $z_i = x_i + iy_i$, in terms of its Cartesian coordinates (x_i, y_i). The interaction is $-e^2 \ln(|z_i - z_j|/a)$ between two

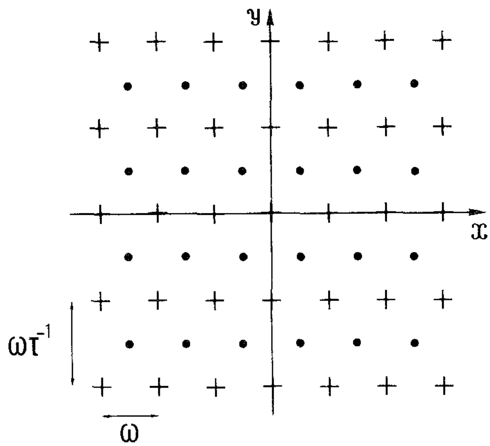

Fig. 1. The Gaudin model. Each site of the sublattice X (crosses) can be occupied by no or one positive particle; each site of the sublattice Y (dots) can be occupied by no or one negative particle.

particles of the same sign and $e^2 \ln(|z_i - z_j|/a)$ between two particles of opposite sign.

2.2. Grand Partition Function

Gaudin showed that, at $\Gamma = 2$, in terms of a fugacity parameter λ, the grand partition function Z has the remarkably simple structure

$$\ln Z = \mathrm{Tr}[\ln(1 + \lambda K)] \tag{2.1}$$

where K is an anti-Hermitic matrix, the elements of which are labeled by the sites and of the form

$$\langle z_i | K | z_j \rangle = \begin{cases} a/(z_i - \bar{z}_j) & \text{if sites } i \text{ and } j \text{ belong to different sublattices} \\ 0 & \text{otherwise} \end{cases}$$

Incidentally, this structure is very general and does not depend on the detail of the lattice geometry (it is only necessary to assume that one of the sublattices is invariant under a symmetry with respect to some axis).

The eigenvalues of K are $\pm i\,|h(\mathbf{k})|$, where $h(\mathbf{k})$ is the Fourier transform of $\langle x_i + iy_i | K | 0 \rangle$ calculated on the lattice. Therefore, the pressure is given by

$$\beta p = \lim_{\Omega \to \infty} \frac{1}{\Omega} \ln Z = \int_{\mathscr{B}} \frac{d^2\mathbf{k}}{(2\pi)^2} \ln[1 + \lambda^2\,|h(\mathbf{k})|^2] \tag{2.2}$$

where Ω is the lattice area and $\mathscr{B}$ the first Brillouin zone. With the geometry of Fig. 1, $\mathscr{B}$ is such that $-\pi/\omega < k_x < \pi/\omega$, $-\pi\tau/\omega < k_y < \pi\tau/\omega$, and

$$\begin{aligned} h(\mathbf{k}) &= \sum_{\substack{m,n \\ (m,n \text{ odd})}} \frac{2a}{\omega(m+in\tau^{-1})} \exp\left[-i\frac{\omega}{2}(k_x m + k_y n\tau^{-1})\right] \\ &= -i\frac{2a}{\omega}\tau\mathbf{K}\left[\operatorname{sd}\left(\frac{\mathbf{K}\omega}{\pi}(k_x+ik_y); k\right)\right]^{-1} \end{aligned} \tag{2.3}$$

where $\operatorname{sd}(u; k) = \operatorname{sn}(u; k)/\operatorname{dn}(u; k)$ is an elliptic function of the complex variable u, with the usual notation of Jacobi; its periods $\mathbf{K}$ and $\tau\mathbf{K}$ and its modulus k are determined by the ratio τ of the sublattice periods. Rescaling λ into $\mu = 2(a/\omega)\,\tau\mathbf{K}\lambda$ and $k_x + ik_y$ into $\xi + i\eta = (\mathbf{K}\omega/\pi)(k_x + ik_y)$, one finds the pressure

$$\beta p = \frac{1}{4\mathbf{K}^2\omega^2}\int_{-\mathbf{K}}^{\mathbf{K}} d\xi \int_{-\tau\mathbf{K}}^{\tau\mathbf{K}} d\eta \, \ln[1 + \mu^2 \,|\operatorname{sd}(\xi + i\eta; k)|^{-2}] \tag{2.4}$$

and the density

$$\rho = \mu\frac{\partial(\beta p)}{\partial\mu} = \frac{1}{2\mathbf{K}^2\omega^2}\int_{-\mathbf{K}}^{\mathbf{K}} d\xi \int_{-\tau\mathbf{K}}^{\tau\mathbf{K}} d\eta \, \frac{\mu^2}{\mu^2 + |\operatorname{sd}(\xi + i\eta; k)|^2} \tag{2.5}$$

The equation of state is determined by (2.4) and (2.5).

2.3. Correlations

The structure (2.1) of $\ln Z$ enables to calculate the correlation functions of any order. One uses the standard method of introducing a fugacity $\lambda(z_i)$, which depends on the site. Then, (2.1) is replaced by

$$\ln Z = \operatorname{Tr}[\ln(1 + A)] \tag{2.6}$$

where the matrix A has elements

$$\langle z_i|\, A \,|z_j\rangle = \lambda(z_i)\langle \tilde{z}_i|\, K \,|\tilde{z}_j\rangle \tag{2.7}$$

with

$$\begin{aligned} \tilde{z}_i &= z_i \qquad \text{if site } i \text{ is a positive one} \\ \tilde{z}_i &= \bar{z}_i \qquad \text{if site } i \text{ is a negative one} \end{aligned}$$

($\bar{z}$ is the complex conjugate of z). Let ρ_s be the maximum density for the particles of one sign (with the geometry of Fig. 1, $\rho_s = \tau/\omega^2$). Using (2.6) and

$$\lambda(z_k) \frac{\partial}{\partial \lambda(z_k)} \langle z_i | \, A \, | z_j \rangle = \delta_{z_k, z_i} \langle z_i | \, A \, | z_j \rangle \tag{2.8}$$

one finds for the density of particles of one sign (number of such particles per unit area)

$$\frac{\rho}{2} = \rho_s \lambda(z_1) \frac{\partial}{\partial \lambda(z_1)} \ln Z \bigg|_{\lambda(z_i) = \lambda} = \rho_s \langle \tilde{z}_1 | \, \frac{\lambda K}{1 + \lambda K} \, | \tilde{z}_1 \rangle \tag{2.9}$$

[It is easy to see that (2.5) is in agreement with (2.9).] Using

$$\frac{\partial}{\partial \lambda(z_k)} \frac{1}{1 + A} = -\frac{1}{1 + A} \frac{\partial A}{\partial \lambda(z_k)} \frac{1}{1 + A} \tag{2.10}$$

one finds for the truncated two-body density

$$\begin{aligned} \rho^{(2)T}(z_1, z_2) &= \rho_s^2 \lambda(z_1) \, \lambda(z_2) \frac{\partial^2}{\partial \lambda(z_1) \, \partial \lambda(z_2)} \ln Z \bigg|_{\lambda(z_i) = \lambda} \\ &= -\rho_s^2 \langle \tilde{z}_1 | \, \frac{\lambda K}{1 + \lambda K} \, | \tilde{z}_2 \rangle \langle \tilde{z}_2 | \, \frac{\lambda K}{1 + \lambda K} \, | \tilde{z}_1 \rangle \end{aligned} \tag{2.11}$$

By successive iterations, one finds for the truncated n-body density

$$\rho^{(n)T}(z_1, z_2, \ldots, z_n) = (-)^{n+1} \sum_{(i_1 i_2 \cdots i_n)} B(z_{i_1}, z_{i_2}) \cdots B(z_{i_n}, z_{i_1}) \tag{2.12}$$

where

$$B(z_i, z_j) = \rho_s \langle \tilde{z}_i | \, \frac{\lambda K}{1 + \lambda K} \, | \tilde{z}_j \rangle \tag{2.13}$$

and where the summation runs over all cycles $(i_1 i_2 \cdots i_n)$ built with $\{1, 2, \ldots, n\}$.

This cyclic structure is similar to that in the case of the two-dimensional one-component plasma[7] at $\Gamma = 2$.

The matrix elements (B functions) in (2.9), (2.11), and (2.12) can be obtained by using the eigenvectors (plane-wave-like) and eigenvalues $\pm i \, |h(\mathbf{k})|$ of the matrix K. Instead of z_i, it may be convenient to use the notation $\mathbf{r}_i = (x_i, y_i) = (\operatorname{Re} z_i, \operatorname{Im} z_i)$ and a subscript e_i which is $+$ $(-)$ if the site at $\mathbf{r}_i$ belongs to the positive (negative) sublattice. One finds for the B functions

$$B_{++}(\mathbf{r}_i, \mathbf{r}_j) = B_{--}(\mathbf{r}_i, \mathbf{r}_j)$$

$$= \int_{\mathscr{B}} \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{\lambda^2}{\lambda^2 + |f(\mathbf{k})|^2} \exp[i\mathbf{k}\cdot(\mathbf{r}_i - \mathbf{r}_j)] \tag{2.14}$$

$$B_{+-}(\mathbf{r}_i, \mathbf{r}_j) = -\overline{B_{-+}(r_j, r_i)}$$

$$= \int_{\mathscr{B}} \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{\lambda \bar{f}(\mathbf{k})}{\lambda^2 + |f(\mathbf{k})|^2} \exp[i\mathbf{k}\cdot(\mathbf{r}_i - \mathbf{r}_j)] \tag{2.15}$$

where $f(\mathbf{k}) = 1/h(\mathbf{k})$ [see (2.3)]. When convenient, we shall use the notation

$$\rho^{(n)T}_{e_1 e_2 \cdots e_n}(\mathbf{r}_1, \mathbf{r}_2,..., \mathbf{r}_n)$$

for $\rho^{(n)T}(z_1, z_2,..., z_n)$.

Since B_{++} is real and symmetrical, $\rho^{(2)T}$ is real, negative; on the other hand, $\rho^{(n)T}_{+-}$ is real, positive.

In the limiting cases considered in Sections 3 and 4, it will be shown that the truncated densities (2.12) have an exponential decay for large space separations. We conjecture that this exponential decay holds in general.

2.4. Sum Rules

The correlation functions are expected to obey a variety of sum rules.

Of very general validity is the compressibility sum rule, which reads here

$$\frac{2}{\rho_s} \sum_j \rho^{(2)T}(\mathbf{r}_i, \mathbf{r}_j) = \lambda \frac{\partial \rho}{\partial \lambda} - \rho = \rho \frac{\partial \rho}{\partial(\beta p)} - \rho \tag{2.16}$$

This rule can be verified by using the structure of (2.9) and (2.11), but actually it is a consequence of the expressions of ρ and $\rho^{(2)T}$ as derivatives of $\ln Z$ with respect to $\lambda(z_i)$. The compressibility sum rule can readily be generalized to higher order truncated densities.

Another class of sum rules is associated with the existence of screening, a fundamental property of a conductor. Checking that these sum rules are satisfied (and they are!) provides a test that, at $\Gamma = 2$, the system is indeed in its conductive phase (the existence of exponential decay for the correlations being another indication).

A first screening rule states that a particle of the gas is screened by the other charges; thus, the charge–charge correlation function obeys

$$\frac{1}{\rho_s}\left[\sum_{\substack{j\\(e_j=e_i)}} \rho^{(2)T}_{++}(\mathbf{r}_i,\mathbf{r}_j) - \sum_{\substack{j\\(e_j\neq e_i)}} \rho^{(2)T}_{+-}(\mathbf{r}_i,\mathbf{r}_j)\right] = -\frac{\rho}{2} \tag{2.17}$$

This rule can readily be verified by expressing ρ and $\rho^{(2)T}$ in terms of the B functions (2.14) and (2.15).

The assumption that an *external* test charge is screened by the gas results in a second screening rule, the Stillinger–Lovett rule,[8] which involves the second moment of the charge–charge correlation function:

$$\frac{2}{\rho_s}\left[\sum_{\substack{j\\(e_j=e_i)}} \rho^{(2)T}_{++}(\mathbf{r}_i,\mathbf{r}_j)(\mathbf{r}_j-\mathbf{r}_i)^2 - \sum_{\substack{j\\(e_j\neq e_i)}} \rho^{(2)T}_{+-}(\mathbf{r}_i,\mathbf{r}_j)(\mathbf{r}_j-\mathbf{r}_i)^2\right]$$

$$= -\frac{2}{\pi\beta e^2} = -\frac{1}{\pi} \tag{2.18}$$

Introducing again the B functions (2.14) and (2.15), we can write the left-hand-side of (2.18) as

$$I = -2\int_{\mathscr{B}} \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{\lambda^2 \nabla_{\mathbf{k}} \bar{f}(\mathbf{k})\cdot\nabla_{\mathbf{k}} f(\mathbf{k})}{(\lambda^2+|f(\mathbf{k})|^2)^2} \tag{2.19}$$

where $f(\mathbf{k}) = 1/h(\mathbf{k})$ is defined by (2.3) and therefore is an analytic function of $k_x + ik_y$. Using for $f(\mathbf{k})$ the simpler notation $f(k_x+ik_y)$, we obtain

$$I = -2\int_{\mathscr{B}} \frac{dk_x\,dk_y}{(2\pi)^2} \frac{2\lambda^2\,|f'|^2}{(\lambda^2+|f|^2)^2} \tag{2.20}$$

We can now make a change of variables from (k_x, k_y) to $(P = \operatorname{Re} f$, $Q = \operatorname{Im} f)$. The Jacobian of this transformation is $|f'|^2$, and, up to a multiplicative constants, f is the elliptic function sd, which provides a one-to-one mapping from the domain $\mathscr{B}$ onto the whole complex plane. Therefore, (2.20) simply becomes

$$I = -2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dP\,dQ}{(2\pi)^2}\frac{2\lambda^2}{(\lambda^2+P^2+Q^2)^2} = -\frac{1}{\pi} \tag{2.21}$$

which proves (2.18).

Moreover, since

$$(\partial/\partial k_x)\, f(k_x+ik_y) = -i(\partial/\partial k_y)\, f(k_x+ik_y)$$

if in (2.18) we split $(\mathbf{r}_j-\mathbf{r}_i)^2$ into $(x_j-x_i)^2$ and $(y_j-y_i)^2$, each term gives the same contribution to I. Therefore, the charge–charge correlation function carries no quadrupole moment, even when the lattice is not symmetrical in x and y ($\tau \neq 1$). This absence of quadrupole moment is in agreement with more general screening rules.[9]

The conclusion of the present section is that the lattice model of the Coulomb gas obeys the screening rules of a conductive phase at $\Gamma=2$. The screening, which is a long-range effect, is not spoilt by the short-range forces simulated by the lattice.

2.5. Particle–Hole Symmetry

In a lattice model such as the present one, with a maximum density $2\rho_s$, we expect the densities ρ and $2\rho_s-\rho$ to be related by a particle–hole symmetry. We can show that indeed the correlation functions have very simple symmetry properties under the transformation $\rho \to 2\rho_s-\rho$.

Using the relations

$$\mathrm{sd}(\xi+i\eta) = -\mathrm{sd}(-\xi-i\eta) = \overline{\mathrm{sd}(\xi-i\eta)}$$

we can rewrite ρ and the B functions as integrals on one quadrant of $\mathscr{B}$ [in (2.14) and (2.15) we use $r=(x, y)$ for $\mathbf{r}_j-\mathbf{r}_i$]:

$$\rho = \frac{2}{\mathbf{K}^2\omega^2}\int_0^{\mathbf{K}} d\xi \int_0^{\tau\mathbf{K}} d\eta \, \frac{1}{1+\mu^{-2}\,|\mathrm{sd}(\xi+i\eta)|^2} \tag{2.22}$$

$$B_{++}(\mathbf{r}) = \frac{1}{\mathbf{K}^2\omega^2}\int_0^{\mathbf{K}} d\xi \int_0^{\tau\mathbf{K}} d\eta \, \frac{1}{1+\mu^{-2}\,|\mathrm{sd}(\xi+i\eta)|^2} \times \cos\left(\frac{\pi}{\mathbf{K}\omega}\xi x\right)\cos\left(\frac{\pi}{\mathbf{K}\omega}\eta y\right) \tag{2.23}$$

$$B_{+-}(\mathbf{r}) = \frac{1}{2\mathbf{K}^2\omega^2}\int_0^{\mathbf{K}} d\xi \int_0^{\tau\mathbf{K}} d\eta \, \frac{\mu^{-1}}{1+\mu^{-2}\,|\mathrm{sd}(\xi+i\eta)|^2} \times \left\{\overline{\mathrm{sd}(\xi+i\eta)}\sin\left[\frac{\pi}{\mathbf{K}\omega}(\xi x+\eta y)\right] + \mathrm{sd}(\xi+i\eta)\sin\left[\frac{\pi}{\mathbf{K}\omega}(\xi x-\eta y)\right]\right\} \tag{2.24}$$

In (2.23), x/ω and $\tau y/\omega$ are integers; in (2.24) they are half-integers. The maximum density $2\rho_s = 2\tau/\omega^2$ is obtained as $\mu \to \infty$. Since, under a symmetry with respect to the point $(\mathbf{K}/2)+i(\tau\mathbf{K}/2)$, sd obeys the relation

$$\mathrm{sd}(\mathbf{K}+i\tau\mathbf{K}-u) = \frac{i}{kk'}\frac{1}{\mathrm{sd}(u)} \tag{2.25}$$

[k' is the complementary modulus $(1-k^2)^{1/2}$], it is readily seen that, under the transformation $\mu \to 1/kk'\mu$, ρ is transformed into $2\rho_s - \rho$, while

$$B_{++}(\mathbf{r}; \mu) = (-1)^{(x+\tau y)/\omega + 1} B_{++}(\mathbf{r}; 1/kk'\mu) \tag{2.26}$$

$$B_{+-}(\mathbf{r}; \mu) = i(-)^{(x+\tau y)/\omega} \overline{B_{+-}(\mathbf{r}; 1/kk'\mu)} \tag{2.27}$$

Therefore, the n-body truncated densities are simply unchanged if n is even or change sign if n is odd, as expected.

The lattice is exactly half-filled ($\rho = 2\rho_s - \rho$) at the symmetry point $\mu^2 = 1/kk'$. For this value, one finds that B_{++} and $\rho^{(2)T}_{++}$ vanish on the sublattice where $(x+\tau y)/\omega$ is even, while $\rho^{(2)T}_{++}$ remains negative on the other sublattice where $(x+\tau y)/\omega$ is odd. This means that like particles show some (imperfect) tendency to order themselves in a superlattice, occuping one site out of two.

3. PARALLEL LINE MODEL

3.1. Definition of the Model

It would be more convenient to have a solvable model in terms of elementary functions only. In Gaudin's lattice model, the appearance of an elliptic function reflects the double periodicity of the lattice in both the x and y directions. We found that this elliptic function is replaced by a trigonometric function if we slightly alter the model in such a way that it has a finite periodicity in only one direction. This simplification can be made without bringing about any divergence, by considering a model in which the charges of the same sign are constrained to move on lines which are parallel to the y axis, with the positively and negatively charged lines alternating and separated by the distance $\omega/2$. On this more tractable model, we are able to study in some detail the asymptotic behavior of the correlation functions and its dependence on the density.

3.2. Pressure, Density, and Correlation Functions

The line model may be obtained from the lattice model by making the length of the primitive cell edge parallel to the y axis go to zero. The grand partition function still has the remarkable structure (2.1) and the pressure, density, and correlation functions are given by (2.4), (2.5), (2.11), (2.14), and (2.15), where $h(\mathbf{k})$ now reads

$$\begin{aligned} h(\mathbf{k}) &= \sum_{\substack{m \\ (m\ \text{odd})}} \int_{-\infty}^{+\infty} \frac{2}{\omega}\, dy \frac{a}{\frac{1}{2}\omega m + iy} \exp\left[-i\left(\frac{\omega}{2} k_x m + k_y y\right)\right] \\ &= -i\frac{2a}{\omega}\pi\left[\sin\frac{\omega}{2}(k_x + ik_y)\right]^{-1} \end{aligned} \tag{3.1}$$

Then, rescaling λ into $z = 2a2^{1/2}\pi\lambda/\omega$, we find for the pressure

$$\beta p = \int_{-\pi/\omega}^{\pi/\omega} \frac{dk_x}{2\pi} \int_{-\infty}^{+\infty} \frac{dk_y}{2\pi} \ln[1 + z^2(\mathrm{ch}\, \omega k_y - \cos \omega k_x)^{-1}] \qquad (3.2)$$

After integrating over k_x, we find for the density

$$\rho = \frac{4}{\pi\omega^2} z^2 \int_0^{+\infty} d\alpha\, [(\mathrm{ch}\, 2\alpha + z^2)^2 - 1]^{-1/2}$$

$$= \begin{cases} \dfrac{2}{\pi\omega^2} z^2 F\left(\sin^{-1}\left[\dfrac{2}{2+z^2}\right]^{1/2}, \left[1 - \dfrac{z^4}{4}\right]^{1/2}\right) & \text{if } z^2 \leqslant 2 \\ \dfrac{2}{\pi\omega^2} 2F\left(\sin^{-1}\left[\dfrac{z^2}{2+z^2}\right]^{1/2}, \left[1 - \dfrac{4}{z^4}\right]^{1/2}\right) & \text{if } z^2 \geqslant 2 \end{cases} \qquad (3.3)$$

where $F(\varphi, k)$ is the elliptic integral of the first kind; the two expressions (3.3) define the same analytic function of the fugacity z, in different domains. The truncated two-body densities are

$$\rho_{++}^{(2)T}(x, y) = \rho_{--}^{(2)T}(x, y)$$

$$= -\frac{1}{\pi^2} \frac{1}{\omega^4} z^4 \left\{ \int_{-\infty}^{+\infty} d\alpha \frac{1}{[g(\alpha)^2 - 1]^{1/2}} \right.$$

$$\left. \times [G(\alpha)]^{|x/\omega|} \exp\left(i\alpha \frac{2}{\omega} y\right) \right\}^2 \qquad (3.4)$$

and

$$\rho_{+-}^{(2)T}(x, y) = \rho_{-+}^{(2)T}(x, y)$$

$$= \frac{1}{\pi^2} \frac{1}{\omega^4} \frac{z^2}{2} \left| \int_{-\infty}^{+\infty} d\alpha \frac{e^{\alpha}[G(\alpha)]^{1/2} - e^{-\alpha}[G(\alpha)]^{-1/2}}{[g(\alpha)^2 - 1]^{1/2}} \right.$$

$$\left. \times [G(\alpha)]^{|x|/\omega} \exp\left(i\alpha \frac{2}{\omega} y\right) \right|^2 \qquad (3.5)$$

where $g(\alpha) = \mathrm{ch}\, 2\alpha + z^2$ and $G(\alpha) = g(\alpha) - [g(\alpha)^2 - 1]^{1/2}$. For the same reasons as in Section 2, the sum rules are satisfied (in particular, the two-body truncated density carries no quadrupole moment, in spite of the high anisotropy of the model) and the system is in a plasma phase at $\Gamma = 2$.

3.3. Asymptotic Behavior of the Two-Body Correlations

For studying the asymptotic behavior of the correlation functions, we have to distinguish two cases, namely whether **r** is in the direction perpendicular to the lines or not.

(i) In the first case, $\rho_{++}^{(2)T}(x, 0)$ is given by (3.4) with $y=0$; $g^2(\alpha)-1$ does not vanish on the real axis and $G(\alpha)$ is maximum at $\alpha=0$. This maximum governs the asymptotic behavior, which turns out to be a mere exponential decay for every fugacity z:

$$\rho_{++}^{(2)T}(x, 0) \underset{|x|\to\infty}{\simeq} -\frac{1}{2\pi}\frac{z^3}{(z^2+2)^{1/2}}\frac{1}{\omega^3|x|}\exp\left(-\frac{|x|}{l_1(z)}\right) \tag{3.6}$$

where

$$l_1(z)=\frac{\omega}{2}\left|\ln\frac{1}{2}[z-(z^2+2)^{1/2}]^2\right|^{-1}$$

On the other hand,

$$\rho_{+-}^{(2)T}(x, 0) \underset{|x|\to\infty}{\simeq} -\rho_{++}^{(2)T}(x, 0)$$

At high density, the fugacity z goes to infinity, the average distance between particles on the same line is

$$\langle d\rangle = 2/\omega\rho \underset{z\to\infty}{\sim} \pi\omega/\ln(2z^2)$$

whereas

$$l_1(z) \underset{z\to\infty}{\simeq} \omega/[2\ln(2z^2)]$$

Thus the correlation length along the x axis is proportional to the average distance between particles along a given line in the y direction.

(ii) In the second case, that is, when the angle θ between $\mathbf{r}$ and the normal to the lines does not vanish,

$$\rho_{++}^{(2)T}(r, \theta) = -\frac{1}{(\pi\omega^2)^2}z^4\left\{\int_{-\infty}^{+\infty} d\alpha\,\frac{\exp[2rF(\alpha, \theta)/\omega]}{[g(\alpha)^2-1]^{1/2}}\right\}^2$$

$$\rho_{+-}^{(2)T}(r, \theta) = \frac{1}{(\pi\omega^2)^2}\frac{z^2}{2}\left|\int_{-\infty}^{+\infty} d\alpha\,\frac{\exp[2rF(\alpha, \theta)/\omega]}{[g(\alpha)^2-1]^{1/2}}A(\alpha)\right|^2$$

where $A(\alpha)=e^{\alpha}[G(\alpha)]^{1/2}-e^{-\alpha}[G(\alpha)]^{-1/2}$ has no singularity. It can be checked that the contribution of possible saddlepoints on the cut originating from a zero of $g(\alpha)^2-1$ has a faster exponential decay than the contribution of the branch point. Of course we have to take into account only the branch points with a positive imaginary part which are closest to the real axis, as shown on Fig. 2. After calculations, we find correlation functions whose behavior changes when $z^2=2$:

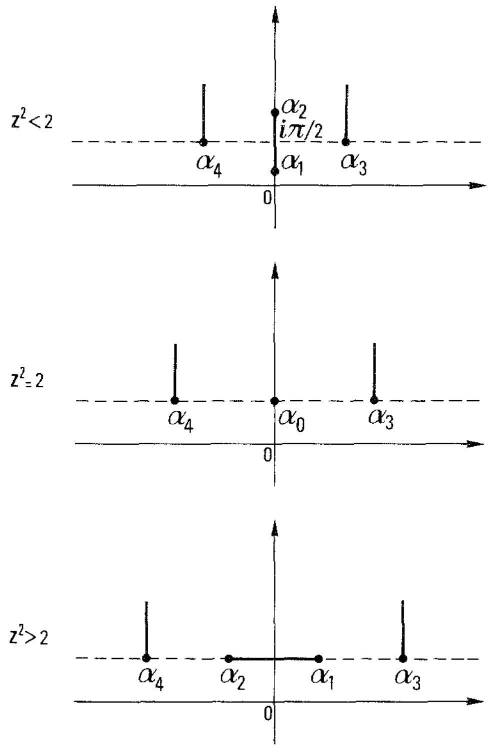

Fig. 2. Location of the branch points and cuts near the real axis, in the α complex plane (in the upper half-plane), for different values of the fugacity. These branch points determine the asymptotic behavior of the correlation functions. For $z^2<2$, only α_1 contributes; α_1 and α_2 get closer to each other as $z^2 \to 2$. For $z^2=2$, α_0, α_3, and α_4 all contribute; α_0 is a pole that results from the coalescence of the branch points α_1 and α_2. For $z^2>2$, α_1, α_2, α_3 and α_4 all contribute.

If $z^2<2$

$$
\begin{aligned}
\rho_{++}^{(2)T}(x, y) &\underset{r\to\infty}{\sim} -\frac{1}{2\pi}\frac{z^3}{(2-z^2)^{1/2}}\frac{1}{\omega^3 r}\exp\left(-\frac{|y|}{l_2(z)}\right)\\
\rho_{+-}^{(2)T}(x, y) &\underset{r\to\infty}{\sim} -\rho_{++}^{(2)T}(x, y)
\end{aligned}
\tag{3.7}
$$

where

$$l_2(z)=\frac{\omega}{2\cos^{-1}(1-z^2)}$$

If $z^2 = 2$

$$\begin{aligned}\rho_{++}^{(2)T}(x, y) \underset{r\to\infty}{\sim} & -4\frac{1}{\omega^4}\exp\left(-\frac{|y|}{l_{2\min}}\right) \\ & \times \left\{1 - \left(\frac{2}{\pi^2}\right)^{1/4} \frac{\cos\{\pi/4 + [(\mathrm{ch}^{-1}3)/\omega]\,|y|\}}{(r/\omega)^{1/2}}\right\}^2 \\ \rho_{+-}^{(2)T}(x, y) \underset{r\to\infty}{\sim} & \, 4\frac{1}{\omega^4}\exp\left(-\frac{|y|}{l_{2\min}}\right) \\ & \times \left\{1 + \left(\frac{2}{\pi^2}\right)^{1/4} \frac{\sin\{\pi/4 + [(\mathrm{ch}^{-1}3)/\omega]\,|y|\}}{(r/\omega)^{1/2}}\right\}^2\end{aligned} \tag{3.8}$$

where $l_{2\min} = \omega/2\pi$. We still have

$$\lim_{r\to\infty} [\rho_{+-}^{(2)T}(x, y)/\rho_{++}^{(2)T}(x, y)] = -1$$

but now the oscillatory corrections to $[-\rho_{++}^{(2)T}]^{1/2}$ and $[\rho_{+-}^{(2)T}]^{1/2}$ are shifted by a phase factor $\pi/2$.

If $z^2 > 2$

$$\begin{aligned}\rho_{++}^{(2)T}(x, y) \underset{r\to\infty}{\sim} & -\frac{2}{\pi} z^3 \frac{1}{\omega^3 r}\exp\left(-\frac{|y|}{l_{2\min}}\right) \\ & \times \left\{\frac{1}{(z^2-2)^{1/4}}\cos\left[\frac{\mathrm{ch}^{-1}(z^2-1)}{\omega}\,|y| - \frac{\pi}{4}\right]\right. \\ & \left. - \frac{1}{(z^2+2)^{1/4}}\cos\left[\frac{\mathrm{ch}^{-1}(z^2+1)}{\omega}\,|y| + \frac{\pi}{4}\right]\right\}^2 \\ \rho_{+-}^{(2)T}(x, y) \underset{r\to\infty}{\sim} & \frac{2}{\pi} z^3 \frac{1}{\omega^3 r}\exp\left(-\frac{|y|}{l_{2\min}}\right) \\ & \times \left\{\frac{1}{(z^2-2)^{1/2}}\sin^2\left[\frac{\mathrm{ch}^{-1}(z^2-1)}{\omega}\,|y| - \frac{\pi}{4}\right]\right. \\ & \left. + \frac{1}{(z^2+2)^{1/2}}\sin^2\left[\frac{\mathrm{ch}^{-1}(z^2+1)}{\omega}\,|y| + \frac{\pi}{4}\right]\right\}\end{aligned} \tag{3.9}$$

When $z^2 < 2$ the correlation length l_2 is a decreasing function of the density. At $z^2 = 2$, l_2 reaches a minimum value $\omega/2\pi$ which only depends on the spacing between the lines, and l_2 keeps that same value at higher densities; thus, the decay is sensitive to a microscopic feature, namely the period of the line array. Furthermore, when $z^2 < 2$ the correlation function behavior is a mere exponential decay, whereas at $z^2=2$ an oscillatory correction appears [and the decay is $\exp(-|y|/l)$ instead of $r^{-1}\exp(-|y|/l)$]; for higher densities, $\rho_{++}^{(2)T}(x, y)$ [$\rho_{+-}^{(2)T}(x, y)$] has an oscillatory exponential decay with four (two) frequencies $\Omega(z)=(2/\omega)\,\mathrm{ch}^{-1}(z^2-1)$, $\Omega'(z)=(2/\omega)\,\mathrm{ch}^{-1}(z^2+1)$, $(\Omega+\Omega')/2$, and $(\Omega'-\Omega)/2$ (Ω and Ω'), the first three being increasing functions of the density. When $z \gg 1$,

$$\Omega(z) \sim \Omega'(z) \sim (2/\omega)\ln(2z^2) \sim 2\pi/\langle d\rangle$$

When the average distance $\langle d\rangle$ between particles on the same line becomes negligible compared to the spacing ω between the lines, the periodicity of the oscillations involves the average distance $\langle d\rangle$ (which also is proportional to the correlation length l_1 along the normal to the lines). There is a tendency to a kind of damped crystal ordering, not unlike what happens in a related one-dimensional model.[10] Incidentally, we recover this one-dimensional model (particles of the same sign on one line) by taking the limit $z\to\infty$ at fixed $\langle d\rangle$ and y; this implies $\omega\to\infty$, i.e., the other lines go away to infinity. Then, the limit of the two-body truncated linear density is

$$\lim_{z\to\infty}\,[\omega^2\rho_{++}^{(2)T}(0, y)] = -\frac{1-\cos(2\pi y/\langle d\rangle)}{2\pi^2 y^2} \tag{3.10}$$

and this indeed is the one-dimensional result.

Finally let us consider the low-density limit (which will be further studied in the next section). Then z and ω go to zero in such a way that $l_0(z)=2^{-3/2}\omega/z$ remains finite. If we first make r go to infinity and then z to zero, we obtain

$$\begin{aligned} |\rho_{\substack{++\\+-}}^{(2)T}(x, y)| &\sim \frac{1}{8\pi}\frac{1}{(2l_0)^3}\exp\left(-\frac{|y|}{l_0}\right) \\ |\rho_{\substack{++\\+-}}^{(2)T}(x, 0)| &\sim \frac{1}{8\pi}\frac{1}{(2l_0)^3}\exp\left(-\frac{|x|}{l_0}\right) \end{aligned} \tag{3.11}$$

whereas if we first make z go to zero and then r to infinity,

$$|\rho_{\substack{++\\+-}}^{(2)T}(x, y)| \sim \frac{1}{8\pi}\frac{1}{(2l_0)^3}\exp\left(-\frac{r}{l_0}\right) \tag{3.12}$$

Thus the two limits do not commute and only the first procedure keeps a memory of the anisotropy of the initial discrete model.

The motion of the branch points discussed with regard to Fig. 2 has some similarity with a possible mechanism[11] for the appearance of oscillations in the correlation function of a three-dimensional plasma.

4. THE CONTINUUM LIMIT

As the lattice spacing ω goes to zero, the lattice model is expected to go over to a simpler continuum model, in which a particle can occupy any position in the plane. The dimensionless parameter that describes the density is $\rho\omega^2$; thus, the small-ω limit is also the low-density limit considered by Gaudin.[6] In terms of the dimensionless fugacity μ, it is a small-μ limit.

4.1. Density and Pressure

For studying the continuum limit, we introduce a rescaled fugacity $m = \pi\mu/\mathbf{K}\omega = 2\pi a \tau \lambda/\omega^2$; m has the dimensions of an inverse length, and m^{-1} will be found to be a correlation length. It will be convenient to study the limit $\omega \to 0$, $\mu \to 0$ at a finite value of m. The density (2.5) can also be written as

$$\rho = 2 \int_{\mathscr{B}} \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{\mu^2}{\mu^2 + |\mathrm{sd}((\mathbf{K}\omega/\pi)(k_x + ik_y))|^2} \tag{4.1}$$

At small μ, the integral in (4.1) is dominated by the small values of $|\mathrm{sd}|^2$, which are obtained for $\omega\mathbf{k}$ small; since, for small u, $\mathrm{sd}\, u \sim u$,

$$\rho \sim 2 \int_{k \lesssim \omega^{-1}} \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{m^2}{m^2 + k^2} \tag{4.2}$$

For obtaining a finite result, it is necessary to keep in (4.2) the cutoff in $\mathbf{k}$ space at $k \sim 1/\omega$. The appearance of a divergence when ω is strictly zero is of course related to the collapse of pairs discussed in the Introduction. Keeping the cutoff, one finds

$$\rho = \frac{m^2}{2\pi} \left[\ln \frac{C}{m^2\omega^2} + o(1) \right] \tag{4.3}$$

where C is a numerical constant. A result of the form (4.3) can also be obtained starting from the line model. It should be noted that, in the present low-density limit, $\rho\omega^2 \to 0$ and $\rho m^{-2} \to \infty$, namely the average interparticle distance becomes large compared to the lattice spacing (low density), but it becomes small compared to the correlation length (despite its tendency to collapse into bound pairs, the system manages to have screening properties, but many particles are needed for achieving the screening of a given particle).

The pressure can be obtained either from (2.4) or by integrating $\rho = m\, \partial(\beta p)/\partial m$, with the result

$$\beta p = \frac{m^2}{4\pi} \left[\ln \frac{C}{m^2 \omega^2} + 1 + o(1) \right] \tag{4.4}$$

Both ρ and βp depend on the cutoff ω and diverge if $\omega \to 0$ at fixed m. However, the ratio $\beta p/\rho$ has a well-defined limit

$$\beta p/\rho \to 1/2 \tag{4.5}$$

This result (4.5), at $\Gamma = 2$, is in agreement with the equation of state for point particles at $\Gamma < 2$, obtained from a scaling argument[1]

$$\beta p = \rho(1 - \Gamma/4) \tag{4.6}$$

4.2. Correlations

In contrast to the one-body density (4.3), the n-body truncated densities ($n \geqslant 2$) have the very remarkable property of going to well-defined limits as the cutoff ω vanishes for a fixed m. The B functions (2.14) and (2.15) simply become, in terms of integrals on the *whole* $\mathbf{k}$ plane,

$$\begin{aligned} B_{++}(\mathbf{r}) &= \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{m^2}{m^2 + k^2} \exp(-i\mathbf{k} \cdot \mathbf{r}) \\ &= \frac{m^2}{2\pi} K_0(mr) \end{aligned} \tag{4.7}$$

$$\begin{aligned} B_{+-}(\mathbf{r}) &= \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{m(-ik_x - k_y)}{m^2 + k^2} \exp(-i\mathbf{k} \cdot \mathbf{r}) \\ &= -\frac{m^2}{2\pi} [\exp(-i\theta)]\, K_1(mr) \end{aligned} \tag{4.8}$$

where $\mathbf{r} = \mathbf{r}_j - \mathbf{r}_i$ and θ is the polar angle of $\mathbf{r}$; K_0 and K_1 are Bessel functions. The two-body truncated densities are

$$\rho_{++}^{(2)T}(r) = -\left(\frac{m^2}{2\pi}\right)^2 K_0^2(mr) \tag{4.9}$$

$$\rho_{+-}^{(2)T}(r) = \left(\frac{m^2}{2\pi}\right)^2 K_1^2(mr) \tag{4.10}$$

They have exponential asymptotic behaviors as $r \to \infty$:

$$-\rho_{++}^{(2)T}(r) \sim \rho_{+-}^{(2)T}(r) \sim \frac{m^3}{8\pi r} e^{-2mr} \tag{4.11}$$

and therefore m^{-1} is indeed a correlation length. Higher order densities can also be obtained in terms of the B functions (4.7) and (4.8), by using (2.12).

The simple forms (4.7) and (4.8) could also have been obtained by using the equivalence between the Coulomb gas at $\Gamma = 2$ and a free Fermi field[2] (see Appendix B).

The logarithmic behavior of $\rho_{++}^{(2)T}(r)$ at small r can be understood as a limiting case of the $r^{2-\Gamma}$ behavior[12,13] which occurs for $1 < \Gamma < 2$.

4.3. Charged, Hard Disks

We expect the properties of a Coulomb gas at low density to be insensitive to the details of the short-distance cutoff. Therefore, the results in this section, starting from a lattice model, should also be applicable to a system of charged, hard disks of diameter σ in the limit where $\rho\sigma^2$ is small. Only the constant C in (4.3) and (4.4) will depend on the details of the cutoff. Thus, we consider a system of charged, hard disks at $\Gamma = 2$ and at low density, with the purpose of computing the internal energy and the specific heat.

We make the heuristic assumption that the leading terms in the low-density expressions of the density, energy, and specific heat are correctly given by using the zero-density correlation functions (4.9) and (4.10) outside the hard core; of course, inside the hard core, the two-body untruncated densities just vanish.

Our control parameter is the fugacity m, in terms of which the one-body density can be obtained by using the perfect-screening rule (2.18), which reads here

$$\rho = 2 \int_\sigma^\infty 2\pi \, dr \, r[\rho_{+-}^{(2)T}(r) - \rho_{++}^{(2)T}(r)] \tag{4.12}$$

Using (4.9) and (4.10), we find

$$\rho = \frac{m^2}{\pi} \left[\ln \frac{2}{\sigma m} - \gamma + o(1) \right] \tag{4.13}$$

[2] This equivalence has been nicely explained by Nicolaides.[14] However, Nicolaides has defined renormalized quantities whose physical meaning is not apparent, at least to us.

where $\gamma = 0.5772$ is Euler's constant. Equation (4.13) is indeed of the form (4.3), with σ now playing the role of ω. The pressure (4.4) can be written

$$\beta p = \rho/2 + m^2/4\pi + o(1) \tag{4.14}$$

It can be checked that (4.9), (4.10), (4.13), and (4.14) satisfy the virial theorem

$$\beta p = \tfrac{1}{2}\rho + \pi\sigma^2[\rho_{+-}^{(2)T}(\sigma) + \rho_{++}^{(2)T}(\sigma) + \rho^2/2] \tag{4.15}$$

and the compressibility sum rule

$$\rho \frac{\partial \rho}{\partial(\beta p)} = \rho + 2 \int d^2\mathbf{r}\; [\rho_{+-}^{(2)T}(r) + \rho_{++}^{(2)T}(r)] \tag{4.16}$$

up to the requested order m^2. We also find

$$h_{++}(\sigma) = (2/\rho)^2\, \rho_{++}^{(2)T}(\sigma) = -1 + o(1) \tag{4.17}$$

as expected because of the Coulomb repulsion.

The excess internal energy per particle is

$$u = (1/\rho) \int_\sigma^\infty 2\pi\, dr\, r[\rho_{+-}^{(2)T}(r) - \rho_{++}^{(2)T}(r)]\, e^2 \ln(r/a) \tag{4.18}$$

Using (4.9), (4.10), and (4.13), we find

$$u = \tfrac{1}{4}e^2[\ln(2\sigma/ma^2) - \gamma + o(1)] \tag{4.19}$$

The excess specific heat at constant volume, per particle, is

$$c_v = -\frac{\beta^2}{\rho}\left\{\left(\frac{\partial(\rho u)}{\partial\beta}\right)_m + \frac{m\{[\partial(\rho u)/\partial m]_\beta\}^2}{(\partial\rho/\partial m)_\beta}\right\} \tag{4.20}$$

The second term in (4.20) appears because, at constant volume, the fugacity m varies with the temperature; this term can be computed by using (4.13) and (4.19). As to the first term in (4.20), it can be expressed as a fluctuation of the potential energy, which can be written in terms of the truncated densities:

$$-\frac{1}{e^4}\left(\frac{\partial(\rho u)}{\partial\beta}\right)_m$$

$$= \int d^2\mathbf{r}\; [\rho_{+-}^{(2)T}(r) + \rho_{++}^{(2)T}(r)] \left(\ln\frac{r}{a}\right)^2$$

$$+2\int d^2\mathbf{r}_2\, d^2\mathbf{r}_3\, [\rho^{(3)T}_{+++}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)+\rho^{(3)T}_{+--}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)$$

$$-2\rho^{(3)T}_{++-}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)]\ln\frac{r_{12}}{a}\ln\frac{r_{13}}{a}$$

$$+\frac{1}{2}\int d^2\mathbf{r}_2\, d^2\mathbf{r}_3\, d^2\mathbf{r}_4 \Bigg\{\rho^{(4)T}_{++++}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)+\rho^{(4)T}_{++--}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)$$

$$+2\rho^{(4)T}_{+-+-}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)-4\rho^{(4)T}_{+++-}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)+4\Bigg[\rho^{(2)T}_{++}(\mathbf{r}_1,\mathbf{r}_3)$$

$$-\rho^{(2)T}_{+-}(\mathbf{r}_1,\mathbf{r}_3)+\frac{\rho}{2}\delta(\mathbf{r}_1-\mathbf{r}_3)\Bigg]\Bigg[\rho^{(2)T}_{++}(\mathbf{r}_2,\mathbf{r}_4)-\rho^{(2)T}_{+-}(\mathbf{r}_2,\mathbf{r}_4)$$

$$+\frac{\rho}{2}\delta(\mathbf{r}_2-\mathbf{r}_4)\Bigg]\Bigg\}\ln\frac{r_{12}}{a}\ln\frac{r_{34}}{a} \tag{4.21}$$

By inspection, it can be seen that only the first integral in (4.21) gives a contribution to c_v that is not $o(1)$. Using (4.9) and (4.10), one finally finds

$$c_v=\frac{1}{6}\left(\ln\frac{2}{\sigma m}-\gamma\right)^2-\frac{1}{4}\left(\ln\frac{2}{\sigma m}-\gamma\right)-\frac{1}{8}+O\left(\frac{1}{\ln \sigma m}\right) \tag{4.22}$$

The neglected terms in (4.19) essentially come from the hard core of a third particle, and they are expected to be of order $\rho\sigma^2$ (times, perhaps, some logarithm), while in (4.22) we have neglected terms of order $(\ln \sigma m)^{-1}$. These theoretical low-density results can be compared to numerical values obtained in a recent simulation.[15] This is done in Table I. It should be noted that, even for the lowest densities used in this simulation, although $\rho\sigma^2$ and $m^2\sigma^2$ are very small ($\sim 10^{-3}$), the logarithm

Table I. Excess Internal Energy per Particle and Excess Specific Heat per Particle[a]

$\pi\rho\sigma^2/4$	$m^2\sigma^2/4$ (theory)	u/e^2 Theory	u/e^2 Simulation	c_v Theory	c_v Simulation
5×10^{-4}	1.2806×10^{-4}	0.976	0.972(5)	1.4	2.0(2)
5×10^{-3}	1.970×10^{-3}	0.634	0.639(2)	0.3	0.9(1)

[a] The zero of energy is determined by the choice $a=\sigma$.

of $2/m\sigma$ is not much larger than 1. This explains why the energy (4.19) is in excellent agreement with the simulation results, while the specific heat (4.22) is not so good for these finite densities.

5. CONCLUSION

For a system of point particles (no hard core), a collapse occurs for $\Gamma \geqslant 2$, and the internal energy per particle becomes divergent. However, the specific heat per particle is expected[1] to remain finite for $\Gamma > 2$, and is given by an independent pair description; its excess value is

$$c_v = 1 + \tfrac{1}{2}(1 - 2/\Gamma)^{-2} \tag{5.1}$$

A similar expression should also be valid, asymptotically, as $\Gamma \to 2^-$. Thus, the specific heat has an infinite peak at $\Gamma = 2$.

When a hard core is introduced, no collapse occurs, and $\Gamma = 2$ is no longer a singular point. However, at low density ($\rho\sigma^2$ small), the specific heat still has a (now finite) peak near $\Gamma = 2$ [increasing $\rho\sigma^2$ rounds this peak more and more, are seen in the numerical simulation[15]]. In the present paper, we have computed c_v at $\Gamma = 2$ for small $\rho\sigma^2$. The leading terms of u and c_v can be obtained very simply by the same independent pair description that leads to (5.1); one just writes the partition function of a pair as

$$\int_{\sigma < r < m^{-1}} d^2\mathbf{r} \exp[-\Gamma \ln(r/a)]$$

where an upper cutoff has been made at some correlation length m^{-1}. This pair model also gives the equation of state $\beta p = \rho/2$.

Nevertheless, at $\Gamma = 2$, the system is a conductor. The correlation functions obey the screening sum rules and they have an exponential decay at large distances.

Thus, at $\Gamma = 2$, the two-dimensional Coulomb gas exhibits complementary features. It has the energetics of paired particles and the correlations of an ionized system.

APPENDIX A

We come back to the lattice model, now in the weak coupling limit $\Gamma \to 0$. For dealing with this case we build a kind of Debye–Hückel theory, suitably modified, however, for taking into account the existence of a maximum total number density $2\rho_s$.

In terms of a fugacity λ, the average occupation number of a site is

$$n = \lambda/(1+\lambda) \tag{A.1}$$

If now a positive particle is fixed at the origin, the average electrostatic potential on the site at $\mathbf{r}$ is changed by $\psi(\mathbf{r})$ and, in a mean-field theory approach, the average occupation number of this site is changed by

$$\delta n(\mathbf{r}) = \frac{1}{\lambda^{-1}\exp[\pm\beta e\psi(\mathbf{r})]+1} - \frac{1}{\lambda^{-1}+1} \tag{A.2}$$

[the $\pm$ sign in (A.2) must be taken as $+$ $(-)$ for a site on the positive (negative) sublattice]. Linearizing (A.2) with respect to $e\psi$, we find

$$\pm\delta n(\mathbf{r}) = n(1-n)\,\beta e\psi(\mathbf{r}) \tag{A.3}$$

The following is the same as in the standard Debye–Hückel theory. Since the correlation length will turn out to be large compared to the lattice spacing, we can write the same Poisson equation as in the continuum

$$\Delta\psi(\mathbf{r}) = -2\pi[e\,\delta(\mathbf{r}) \pm e\rho_s\,\delta n(\mathbf{r})] \tag{A.4}$$

and we obtain from (A.3) and (A.4) the usual equation

$$[\Delta + \kappa^2]\,\psi(\mathbf{r}) = -2\pi e\,\delta(\mathbf{r}) \tag{A.5}$$

except that the Debye length κ^{-1} is now defined by

$$\kappa^2 = 2\pi\beta e^2\rho(1-\rho/2\rho_s) \tag{A.6}$$

This expression clearly exhibits the expected particle–hole symmetry. The correlation length κ^{-1} is minimum for a half-filled lattice.

The solution of (A.5) is the two-dimensional Debye screened potential

$$\psi(r) = eK_0(\kappa r) \tag{A.7}$$

APPENDIX B

The equivalence between the two-dimensional Coulomb gas and a Fermi field model is usually proved through their common equivalence to the sine-Gordon theory.[3–5,14] In this Appendix we directly show the equivalence between the Coulomb gas and a free Fermi field, at $\Gamma = 2$.

The key point is that, in the continuum limit, the matrix K in the grand partition function (2.1) of the Coulomb gas is very simply related to the operator $\gamma^\mu \partial_\mu$ (here represented as $\sigma_x \partial_x + \sigma_y \partial_y$) of a fermion in a two-dimensional Euclidean theory. In the continuum limit, it is convenient to introduce isospinors and to characterize a positive particle located at point z by

$$|z\rangle \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and a negative particle by

$$|z\rangle \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In this notation, the operator K and a related operator $\tilde{K}$ are represented by[3]

$$\langle z_i| \tilde{K} |z_j\rangle = \langle \bar{z}_i| K |\bar{z}_j\rangle = \frac{\sigma_x + i\sigma_y}{2} \frac{a\tau\omega^{-2}}{z_i - z_j} + \frac{\sigma_x - i\sigma_y}{2} \frac{a\tau\omega^{-2}}{\bar{z}_i - \bar{z}_j} \tag{B.1}$$

where the σ are Pauli matrices operating upon the isospinors. By taking the Fourier transform of

$$[-i(\sigma_x k_x + \sigma_y k_y)]^{-1} = i(\sigma_x k_x + \sigma_y k_y)/(k_x^2 + k_y^2)$$

it is straightforward to show that

$$\tilde{K} = 2\pi a\tau\omega^{-2}(\sigma_x \partial_x + \sigma_y \partial_y)^{-1} \tag{B.2}$$

Now, with a fugacity that depends on the position and charge [$\lambda_+(z)$ and $\lambda_-(z)$ are the fugacities of the positive and negative particles, respectively], the grand partition function (2.6) of the Coulomb gas can be written as

$$\begin{aligned} Z &= \mathrm{Det}\{1 + A\} \\ &= \mathrm{Det}\left\{1 + \left[\lambda_+(z_i)\frac{1+\sigma_z}{2} + \lambda_-(z_i)\frac{1-\sigma_z}{2}\right] \langle z_i| \tilde{K} |z_j\rangle\right\} \end{aligned} \tag{B.3}$$

[3] A factor $\tau\omega^{-2}$ appears when discrete sums are replaced by integrals in the continuum limit.

or, using (B.2),

$$Z = \mathrm{Det}\left\{\left[\sigma_x\partial_x + \sigma_y\partial_y + m_+(\mathbf{r})\frac{1+\sigma_z}{2} + m_-(\mathbf{r})\frac{1-\sigma_z}{2}\right]\right.$$
$$\left.\times\ (\sigma_x\partial_x + \sigma_y\partial_y)^{-1}\right\} \tag{B.4}$$

where $m_\pm = 2\pi a\tau\omega^{-2}\lambda_\pm$ is a rescaled fugacity. On the other hand, the partition function of a Euclidean two-dimensional Fermi field with an external coupling that distinguishes the two components of its spinors is

$$Z_{\mathrm{F}}[m] = \int D\psi\, D\bar{\psi}\, \exp\left\{\int d^2\mathbf{r}\, \bar{\psi}\left[\sigma_x\partial_x + \sigma_y\partial_y\right.\right.$$
$$\left.\left. + \frac{1+\sigma_z}{2} m_+(\mathbf{r}) + \frac{1-\sigma_z}{2} m_-(\mathbf{r})\right]\psi\right\}$$
$$= \mathrm{Det}\left[\sigma_x\partial_x + \sigma_y\partial_y + \frac{1+\sigma_z}{2} m_+(\mathbf{r}) + \frac{1-\sigma_z}{2} m_-(\mathbf{r})\right] \tag{B.5}$$

where ψ and $\bar{\psi}$ are two-component Grassmann variables, and therefore the equivalence relation is

$$Z = Z_{\mathrm{F}}[m]/Z_{\mathrm{F}}[0] \tag{B.6}$$

The two components of the spinors of the Fermi field correspond to the two components of the isospinors of the Coulomb gas.

The correlation functions of the Coulomb gas are obtained by taking derivatives of $\ln Z$ with respect to $\lambda_\pm(z)$, while the Green's functions of the Fermi field are obtained by taking derivatives of $\ln Z_{\mathrm{F}}$ with respect to $m_\pm(\mathbf{r})$. The basic B functions (4.6) and (4.7) of the Coulomb gas are the propagators of the Fermi field, and the correlation functions of the Coulomb gas can be expressed in terms of cycles of B functions as the Green's functions of the Fermi field can be expressed through Wick's theorem in terms of cycles of propagators.

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